Extra Practice Problems 6

Here's one final set of practice problems for the second midterm exam. We'll release solutions on Wednesday.

Problem One: Functions

Let $f : \mathbb{N} \to \mathbb{N}$ be a function. We'll say that f is ***linearly bounded*** if $f(n) \leq n$ for all $n \in \mathbb{N}$.

Prove that if $f : \mathbb{N} \to \mathbb{N}$ is linearly bounded and is a bijection, then $f(n) = n$ for all $n \in \mathbb{N}$. *(Hint: You might find induction useful here.)*

Problem Two: The Pigeonhole Principle

Suppose you have a sequence S of rs+1 distinct natural numbers. An ***increasing subsequence*** of S is subsequence of S whose values are in increasing order, and a ***decreasing subsequence*** of S is a subset of S whose values are in decreasing order. The ***Erdős–Szekeres theorem*** states the following: S must either have an increasing subsequence of length at least r+1 or a decreasing subsequence of length s+1.

Suppose that $S = \langle x_1, x_2, \ldots, x_{rs+1} \rangle$. Let's associate with each element x_k of this sequence a pair of natural numbers (I_k, D_k) with the following meaning:

I_k is the length of the longest ***increasing*** subsequence of S whose last element is at position k.
D_k is the length of the longest ***decreasing*** subsequence of S whose last element is at position k.

For example, consider the sequence ⟨40, 20, 10, 30, 50⟩. Then

$(I_1, D_1) = (1, 1)$ $(I_2, D_2) = (1, 2)$ $(I_3, D_3) = (1, 3)$ $(I_4, D_4) = (2, 2)$ $(I_5, D_5) = (3, 1)$

You might want to take a minute to check why these values are correct.

i. Let k be an arbitrary natural number where $1 \leq k \leq rs + 1$. Prove that $I_k \geq 1$ and $D_k \geq 1$.

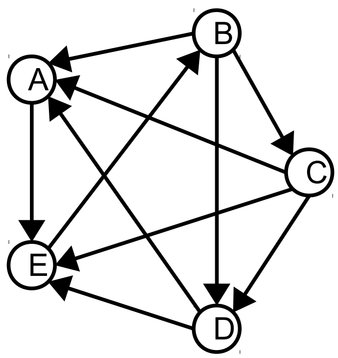
ii. Let j and k be arbitrary natural numbers where $1 \leq j \leq rs + 1$ and $1 \leq k \leq rs + 1$. Prove that if $j \neq k$, then $(I_j, D_j) \neq (I_k, D_k)$. To keep your proof short, we recommend assuming without loss of generality that $j < k$.

iii. Using your results from parts (i) and (ii), prove that any sequence of $rs + 1$ distinct real numbers contains an ascending subsequence of length $r + 1$ or a descending subsequence of length $s + 1$. *(Hint: Proceed by contradiction. If the sequence does not have an ascending subsequence of length $r + 1$ or a decreasing subsequence of length $s + 1$, what do you know about the values of all the (I, D) pairs?)*

Problem Three: Binary Relations

This question explores the interaction between binary relations and tournaments.

Let's quickly refresh a definition. A ***tournament*** is a contest between some number of players in which each player plays each other player exactly once. We assume that no games end in a tie, so each game ends in a win for one of the players.



Here's a new definition to work with. If p is a player in tournament T, then we can define the set $W(p) = \{ x \mid x \text{ is a player in } T \text{ and } p \text{ beat } x \}$. Intuitively, $W(p)$ is the set of all the players that player p beat. For example, in the tournament on the left, $W(B) = \{A, C, D\}$.

Now, let's define a new binary relation. Let T be a tournament. We'll say that $p_1 \sqsubseteq_T p_2$ if $W(p_1) \subseteq W(p_2)$. Intuitively, $p_1 \sqsubseteq_T p_2$ means that p_2 beat every player that p_1 beat, and possibly some additional players.

For example, in the tournament to the left, we have that $D \sqsubseteq_T C$, since $W(D) = \{A, E\}$ and $W(C) = \{A, D, E\}$. Similarly, we know $A \sqsubseteq_T D$ since $W(A) = \{E\}$ and $W(D) = \{A, E\}$.

Prove that if T is any tournament, then $\sqsubseteq_T$ is a partial order over the players in T.

Problem Four: DFAs, NFAs, and Regular Expressions

Let $\Sigma = \{a, b\}$ and consider the following language over Σ:

$$L = \{ w \in \Sigma^* \mid \text{some letter in } w \text{ appears at least four times } \}$$

i. Design an NFA for L.

ii. Write a regular expression for L.

Problem Five: Nonregular Languages

Let $L = \{ w \in \{0, 1, 2\}^* \mid w \text{ contains the same number of copies of the substrings } 01 \text{ and } 10 \}$. This language is similar to the one in Problem Set Five, except that the alphabet is now $\{0, 1, 2\}$ instead of $\{0, 1\}$. Prove that L is not a regular language.

Problem Six: Context-Free Grammars

Let $\Sigma = \{ \land, \lor, \neg, \to, \leftrightarrow, (,), \top, \bot \}$ and let $TRUE_0 = \{ w \in \Sigma^* \mid w \text{ is a propositional formula containing no variables, and that formula is always true} \}$. For example $\top \in TRUE_0$, $\bot \to \top \in TRUE_0$, but $\top \lor \bot \to \bot \notin TRUE_0$. Write a CFG for $TRUE_0$.